UDC 539.3

AXISYMMETRIC DYNAMIC CONTACT PROBLEM FOR A VISCOELASTIC HALF-SPACE IN THE PRESENCE OF ADHESION UNDER THE STAMP

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The steady vibrations of a rigid stamp of circular planform adhering to the surface of a viscoelastic half-space subjected to a harmonic axial force are considered. There are no forces outside the region of contact.

The system of dual integral equations constructed by using the Hankel transform is reduced to a system of singular integral equations regularized according to Vekua [1]. An approximate solution, valid for small vibrations frequencies, is found. Oscillatory behavior of the stresses is proved. Abramov [2] detected a similar fact in the plane case.

The static problem for an elastic medium has been examined by a number of authors [3 – 6]. None of the listed papers detected the oscillatory behavior of the stresses at the edge of the stamp.

1. The complex amplitudes of the displacements of a viscoelastic medium in the presence of axial symmetry satisfy the system of equations

$$(\lambda_* + 2\mu_*)\frac{\partial\Delta}{\partial r} + \mu_*\frac{\partial}{\partial z}(2\omega_\varphi) = -\rho\omega^2 u_r \tag{1.1}$$

$$(\lambda_* + 2\mu_*)\frac{\partial\Delta}{\partial z} - \mu_*\frac{1}{r}\frac{\partial}{\partial r}(2r\omega_\varphi) = -\rho\omega^2 u_z$$

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial}{\partial z}u_z, \qquad 2\omega_\varphi = \frac{\partial}{\partial z}u_r - \frac{\partial}{\partial r}u_z$$

$$\lambda_* = \lambda\left[1 - \int_0^\infty \Lambda(x)e^{-i\omega x}dx\right], \qquad \mu_* = \mu\left[1 - \int_0^\infty M(x)e^{-i\omega x}dx\right]$$

Here ρ is the density of the medium, ω is the vibrations frequency, λ_* and μ_* are the complex moduli.

Let us assume that the medium occupies the half-space $z \leqslant 0$. By using the Hankel transform, we obtain the amplitudes of the displacements in the following form from (1.1):

$$u_z = \int_0^\infty [pe^{pz}A(\omega, s) - \Theta(se^{qz} - pe^{pz})sB(\omega, s)]\, J_0(rs)\, ds \tag{1.2}$$

$$u_r = -\int_0^\infty s\,[e^{pz}A(\omega, s) + \Theta(se^{pz} - qe^{qz})B(\omega, s)]\, J_1(rs)\, ds$$

$$p = (s^2 - k_1^2)^{1/2}, \qquad q = (s^2 - k_2^2)^{1/2}, \qquad \Theta = (k_2^2 - k_1^2)^{-1}$$

$$k_1^2 = \frac{\rho\omega^2}{\lambda_* + 2\mu_*}, \quad k_2^2 = \frac{\rho\omega^2}{\mu_*}, \quad \operatorname{Re} k_1 \geqslant 0, \quad \operatorname{Re} k_2 \geqslant 0$$

Here $A(\omega, s)$ and $B(\omega, s)$ are unknown functions and p and q are branches of the roots which satisfy the conditions $\operatorname{Re} p \geqslant 0$, $\operatorname{Re} q \geqslant 0$. The stress amplitudes are obtained from (1.2) as

$$\frac{1}{2\mu_*}\sigma_z = \int_0^\infty [\varkappa e^{pz}A - \Theta(qse^{qz} - \varkappa e^{pz})sB]\, J_0(rs)\, ds \qquad (1.3)$$

$$\frac{1}{2\mu_*}\tau_{rz} = -\int_0^\infty s\,[pe^{pz}A + \Theta(pse^{pz} - \varkappa e^{qz})B]\, J_1(rs)\, ds$$

$$(\varkappa = s^2 - k_2^2 / 2)$$

The boundary conditions on the surface $z = 0$

$$u_z = f(r),\quad u_r = 0 \quad \text{for } r < R; \quad \sigma_z = \tau_{rz} = 0 \quad \text{for } r \geqslant R$$

should be used to determine A and B. Letting z tend to 0, we obtain a system of dual equations from (1.2) and (1.3)

$$\int_0^\infty [pA - \Theta(s-p)sB]\, J_0(rs)\, ds = f(r) \qquad (1.4)$$

$$r < R$$

$$\int_0^\infty s\,[A + \Theta(s-q)B]\, J_1(rs)\, ds = 0$$

$$\int_0^\infty [\varkappa A - \Theta(qs-\varkappa)sB]\, J_0(rs)\, ds = 0 \qquad (1.5)$$

$$r \geqslant R$$

$$\int_0^\infty s\,[pA + \Theta(ps-\varkappa)B]\, J_1(rs)\, ds = 0$$

2. Analogously to the method developed in [7], let us assume

$$\varkappa A - \Theta(qs-\varkappa)sB = s\int_0^R \varphi_1(t)\cos(ts)\, dt \qquad (2.1)$$

$$pA + \Theta(ps-\varkappa)B = -\int_0^R \varphi_2(t)\sin(ts)\, dt$$

The relationships (2.1) are a linear system of equations in A and B. The functions $\varphi_1(t)$ and $\varphi_2(t)$ are assumed bounded and continuously differentiable in the half-interval $[0, R)$. Substituting (2.1) into (1.3) and using the formulas for differentiation of the Bessel functions, we obtain that at $z = 0$

$$\frac{1}{2\mu_*}\sigma_z = \frac{1}{r}\frac{d}{dr}\int_0^R \varphi_1(t)\left[\int_0^\infty rJ_1(rs)\cos(ts)\, ds\right] dt$$

$$\frac{1}{2\mu_*}\tau_{rz} = -\frac{d}{dr}\int_0^R \varphi_2(t)\left[\int_0^\infty J_0(rs)\sin(ts)\, ds\right] dt$$

From the properties of the Weber-Schafheitlin integrals [8] as well as the properties of $\varphi_1(t)$ and $\varphi_2(t)$ there results that $\sigma_z = \tau_{rz} = 0$ for $r \geqslant R$. The relationships

$$\frac{1}{2\mu_*}\sigma_z = -\frac{1}{r}\frac{d}{dr}\int_r^R \frac{t\varphi_1(t)\,dt}{\sqrt{t^2-r^2}}, \qquad \frac{1}{2\mu_*}\tau_{rz} = -\frac{d}{dr}\int_r^R \frac{\varphi_2(t)\,dt}{\sqrt{t^2-r^2}} \quad (2.2)$$

hold under the stamp. Eliminating A and B from (1.4) and (2.1), we obtain the system

$$\int_0^\infty\int_0^R [g_{11}\varphi_1(t)\cos(ts) + \alpha g_{12}\varphi_2(t)\sin(ts)]\, J_0(rs)\,dtds = \beta f(r) \quad (2.3)$$

$$\int_0^\infty\int_0^R [\alpha g_{21}\varphi_1(t)\cos(ts) + g_{22}\varphi_2(t)\sin(ts)]\, J_1(rs)\,dtds = 0$$

$$g_{11}(k_1, k_2, s) = \frac{\beta k^2 ps}{2D}, \qquad g_{22}(k_1, k_2, s) = \frac{\beta k^2 qs}{2D}$$

$$g_{12}(k_1, k_2, s) = g_{21}(k_1, k_2, s) = -\frac{\gamma s^2(pq - \varkappa)}{D}$$

$$\left(\alpha = \frac{\mu_*}{\lambda_* + 2\mu_*}, \ \beta = \frac{\lambda_* + \mu_*}{\lambda_* + 2\mu_*}, \ \gamma = \frac{\beta}{\alpha}, \ k = k_2, \ D = pqs^2 - \varkappa^2\right)$$

An analysis of the behavior of g_{ij} at infinity shows that the estimate $|1 - g_{ij}| < c(\omega)s^{-2}$ is valid for $|s| \gg |k|$, where $c(0) = 0$. Extracting the unity and using the representations [9]

$$J_0(z) = \frac{2}{\pi}\int_0^{\pi/2} \cos(z\sin\theta)\,d\theta, \quad J_1(z) = \frac{2}{\pi}\int_0^{\pi/2} \sin(z\sin\theta)\sin\theta\,d\theta$$

we reduce the system (2.3) to

$$\int_0^{\pi/2} G(r\sin\theta)\,d\theta = \beta f(r) - \alpha\int_r^R \frac{\varphi_2(t)\,dt}{\sqrt{t^2-r^2}} = g(r) \quad (2.4)$$

$$\int_0^{\pi/2} H(r\sin\theta)\, r\sin\theta\,d\theta = \alpha\int_r^R \frac{t\varphi_1(t)\,dt}{\sqrt{t^2-r^2}} - \alpha\int_0^R \varphi_1(t)\,dt = h(r)$$

$$G(r) = \varphi_1(r) - \frac{2}{\pi}\int_0^R K_{11}(t,r)\varphi_1(t)\,dt - \frac{2\alpha}{\pi}\int_0^R K_{12}(t,r)\varphi_2(t)\,dt$$

$$H(r) = \varphi_2(r) - \frac{2\alpha}{\pi}\int_0^R K_{21}(t,r)\varphi_1(t)\,dt - \frac{2}{\pi}\int_0^R K_{22}(t,r)\varphi_2(t)\,dt$$

$$K_{11} = \int_0^\infty (1 - g_{11})\cos(ts)\cos(rs)\,ds, \quad K_{12} = \int_0^\infty (1 - g_{12})\sin(ts)\cos(rs)\,ds$$

$$K_{21} = \int_0^\infty (1 - g_{21})\cos(ts)\sin(rs)\,ds, \quad K_{22} = \int_0^\infty (1 - g_{22})\sin(ts)\sin(rs)\,ds$$

The $K_{ij}(t, r)$ are evidently continuous functions.

Let us assume that the shape of the stamp is given by a fourth power polynomial in r. If φ_1 and φ_2 are considered known, then each of the equations in the system (2.4) can be considered formally as a Schlömilch equation with a given right side continuously differentiable in the half-interval $[0, R)$. It can be shown that the unique solution of the Schlomilch equation with such a right side continuous in $[0, R)$ is given by the formula

$$G(r)=\frac{2}{\pi}\left[g(0)+r\int_0^{\pi/2} g'(r\sin\theta)\,d\theta\right]= \tag{2.5}$$

$$\frac{2}{\pi}\left[g(0)+r\frac{d}{dr}\int_0^{\pi/2}\frac{g(r\sin\theta)-g(0)}{\sin\theta}\,d\theta\right]$$

Substituting $g(r)$ and $h(r)$ successively into (2.5) and changing the order of integration with respect to t and θ, we obtain

$$G(r)+\frac{\alpha}{\pi}\int_0^R\frac{2t}{t^2-r^2}\varphi_2(t)\,dt=\beta b(r) \tag{2.6}$$

$$H(r)-\frac{\alpha}{\pi}\int_0^R\frac{2r}{t^2-r^2}\varphi_1(t)\,dt=0$$

where $b(r)$ is the inversion of $f(r)$ according to (2.5). The integrals are understood in the principal value sense. Let us note that $\varphi_2(0)=0$, K_{11} and K_{12} are even and K_{21} and K_{22} are odd in r. Continuing $\varphi_1(t)$ evenly to the left and $\varphi_2(t)$ oddly in a continuous way, we obtain a singular system with Cauchy kernel from (2.6)

$$\varphi_1(r)+\frac{\alpha}{\pi}\int_{-R}^R\frac{\varphi_2(t)}{t-r}\,dt-\frac{1}{\pi}\int_{-R}^R K_{11}(t,r)\varphi_1(t)\,dt- \tag{2.7}$$

$$\frac{\alpha}{\pi}\int_{-R}^R K_{12}(t,r)\varphi_2(t)\,dt=\beta b$$

$$\varphi_2(r)-\frac{\alpha}{\pi}\int_{-R}^R\frac{\varphi_1(t)}{t-r}\,dt-\frac{\alpha}{\pi}\int_{-R}^R K_{21}(t,r)\varphi_1(t)\,dt-$$

$$\frac{1}{\pi}\int_{-R}^R K_{22}(t,r)\varphi_2(t)\,dt=0$$

3. Let ω tend to zero. Then (1.2) and (1.3) go over into the known representation of the solution in terms of the biharmonic Love function. Because $\lim K_{ij}=0$ as $\omega\to 0$ for all t and r, the regular part of the system (2.7) vanishes. Therefore, the static problem is described by the characteristic part of (2.7)

$$\varphi_1(r)+\frac{\alpha}{\pi}\int_{-R}^R\frac{\varphi_2(t)}{t-r}\,dt=\beta b \tag{3.1}$$

$$\varphi_2(r)-\frac{\alpha}{\pi}\int_{-R}^R\frac{\varphi_1(t)}{t-r}\,dt=0$$

Let us introduce the analytic functions

$$\Phi_1(\zeta)=\frac{1}{2\pi i}\int_{-R}^R\frac{\varphi_1(t)}{t-\zeta}\,dt,\qquad \Phi_2(\zeta)=\frac{1}{2\pi i}\int_{-R}^R\frac{\varphi_2(t)}{t-\zeta}\,dt$$

By using the Sokhotskii-Plemelj formulas we reduce (3.1) to the two-dimensional problem of a conjugate with piecewise-constant coefficients

$$\Phi^+ = G\Phi^- + g, \ |t| \leqslant R; \ \Phi^+ = \Phi^-, \ |t| > R$$

Here

$$\Phi = \begin{Vmatrix} \Phi_1 \\ \Phi_2 \end{Vmatrix}, \quad G = \begin{Vmatrix} \dfrac{1+\alpha^2}{1-\alpha^2} & \dfrac{-2i\alpha}{1-\alpha^2} \\ \dfrac{2i\alpha}{1-\alpha^2} & \dfrac{1+\alpha^2}{1-\alpha^2} \end{Vmatrix}, \quad g = \begin{Vmatrix} \dfrac{\beta b}{1-\alpha^2} \\ \dfrac{i\alpha\beta b}{1-\alpha^2} \end{Vmatrix}$$

The eigenvalues G are distinct, hence there exists a matrix H such that $H^{-1}GH$ is diagonal. Let us assume

$$\Phi = Hw, \qquad H = \begin{Vmatrix} 1 & -i \\ -i & 1 \end{Vmatrix}, \qquad w = \begin{Vmatrix} w_1 \\ w_2 \end{Vmatrix}$$

$$w_1(\zeta) = \frac{1}{2\pi i}\int_{-R}^{R} \frac{\omega_1(t)}{t-\zeta}\,dt, \qquad w_2(\zeta) = \frac{1}{2\pi i}\int_{-R}^{R} \frac{\omega_2(t)}{t-\zeta}\,dt$$

Then the two-dimensional conjugate problem for Φ reduces to two one-dimensional problems for w

$$w_1^+ = \frac{1-\alpha}{1+\alpha} w_1^- + \frac{\beta b}{2(1+\alpha)}, \qquad w_2^+ = \frac{1+\alpha}{1-\alpha} w_2^- + \frac{i\beta b}{2(1-\alpha)} \tag{3.2}$$

Since $b(r)$ is a polynomial, the solution of (3.2) can then be constructed explicitly.

As an illustration, let us consider a stamp with the flat base $(f(r) = b_0)$. Using the methods developed in [10], we obtain

$$w_1 = \frac{\beta b_0}{2\pi\alpha}[1 - X_1(\zeta)], \qquad w_2 = -\frac{i\beta b_0}{2\pi\alpha}[1 - X_2(\zeta)]$$

$$X_1 = \left(\frac{\zeta - R}{\zeta + R}\right)^{ia}, \qquad X_2 = \left(\frac{\zeta - R}{\zeta + R}\right)^{-ia}, \qquad a = \frac{1}{2\pi}\ln\frac{1+\alpha}{1-\alpha}$$

Evaluating the jumps, we find $\omega_1, \omega_2, \varphi_1, \varphi_2$ by the Sokhotskii-Plemelj formulas

$$\omega_1 = Ax_1(t) = Ae^{ia_*}, \ \omega_2 = iAx_2(t) = iAe^{-ia_*} \tag{3.3}$$

$$\varphi_1 = 2A\cos a_*, \ \varphi_2 = 2A\sin a_*$$

$$\left(A = \frac{\beta b_0}{\pi\sqrt{1-\alpha^2}}, \quad a_* = \frac{1}{2\pi}\ln\frac{1+\alpha}{1-\alpha}\ln\frac{R-t}{R+t}\right)$$

which agrees with the results in [3, 6].

4. Let us assume that $\Lambda(x) \equiv M(x)$. Using the exact solution (3.3), let us regularize the system (2.7).

To simplify the computations, let us reduce (2.7) to the system

$$\omega_1(r) + \frac{\alpha}{\pi i}\int_{-R}^{R} \frac{\omega_1(t)}{t-r}\,dt = \frac{\beta b_0}{\pi} + \frac{1}{2\pi}\int_{-R}^{R} H_{11}(t,r)\,\omega_1(t)\,dt + \tag{4.1}$$

$$\frac{i}{2\pi}\int_{-R}^{R} H_{12}(t,r)\,\omega_2(t)\,dt$$

$$\omega_2(r) - \frac{\alpha}{\pi i}\int_{-R}^{R}\frac{\omega_2(t)}{t-r}\,dt = \frac{i\beta b_1}{\pi} + \frac{i}{2\pi}\int_{-R}^{R}H_{21}(t,r)\,\omega_1(t)\,dt +$$

$$\frac{1}{2\pi}\int_{-R}^{R}H_{22}(t,r)\,\omega_2(t)\,dt$$

by means of the substitution $\varphi = H\omega$. The kernels H_{ij} are connected with the K_{ij} by the relationships

$$\begin{aligned} H_{11} &= K_{11} + K_{22} + i\alpha\,(K_{21} - K_{12}) \\ H_{12} &= K_{22} - K_{11} - i\alpha\,(K_{12} + K_{21}) \\ H_{21} &= K_{11} - K_{22} - i\alpha\,(K_{12} + K_{21}) \\ H_{22} &= K_{11} + K_{22} - i\alpha\,(K_{21} - K_{12}) \end{aligned} \qquad (4.2)$$

Regularizing (4.1) in conformity with Vekua [1], we obtain

$$\omega_1(r) = Ax_1(r) + \frac{1}{2\pi}\int_{-R}^{R}h_{11}(t,r)\,\omega_1(t)\,dt + \frac{i}{2\pi}\int_{-R}^{R}h_{12}(t,r)\,\omega_2(t)\,dt \qquad (4.3)$$

$$\omega_2(r) = iAx_2(r) + \frac{i}{2\pi}\int_{-R}^{R}h_{21}(t,r)\,\omega_1(t)\,dt + \frac{1}{2\pi}\int_{-R}^{R}h_{22}(t,r)\,\omega_2(t)\,dt$$

$$(h_{ij} = w_{ij}^{+} - w_{ij}^{-})$$

The functions w_{ij} are introduced by means of the relationships

$$w_{1n} = \frac{X_1(\zeta)}{2\pi i}\int_{-R}^{R}\frac{H_{1n}(t,\tau)}{(1+\alpha)\,X_1^{+}(\tau)}\,\frac{d\tau}{\tau-\zeta} \qquad (4.4)$$

$$w_{2n} = \frac{X_2(\zeta)}{2\pi i}\int_{-R}^{R}\frac{H_{2n}(t,\tau)}{(1-\alpha)\,X_2^{+}(\tau)}\,\frac{d\tau}{\tau-\zeta}$$

$$(n = 1, 2)$$

The system (4. 3) is a quasi-regular system of Fredholm equations of the second kind.

Let us construct the solution of (4. 3) under the assumption that the parameter $\theta = kR$ is small. It can be shown that the kernels K_{12}, K_{21}, K_{22} are of higher order compared to θ. To estimate K_{11} , let us use the following reasoning. If

$$\mathrm{Re}\,(s^2 - k_{1,2}^2)^{1/2} \geqslant 0, \quad \mathrm{Re}\,(s^2 - \bar{k}_{1,2}^2)^{1/2} \geqslant 0,$$

then

$$\mathrm{Re}\,K_{11} = \frac{1}{4}\,\mathrm{Re}\int_0^\infty [2 - g_{11}(k_1,k_2,s) - g_{11}(\bar{k}_1,\bar{k}_2,s)]\,(e^{i|t-r|s} + e^{i|t+r|s})\,ds$$

In the first quadrant the integrand has a first order pole and the branch point $\bar{k}_1$, $\bar{k}_2$. It takes on real values on the imaginary axis. Taking the contour of integration indicated in Fig. 1 and using the estimate $|\,1 - g_{11}\,| < c\,(\omega)\,s^{-2}$, we obtain that the real part of the kernel K_{11} equals the real part of the sum of the residue multiplied by $2\pi i$ and the integrals over the edges of the slit. Retaining first order terms in this sum, we obtain

$$\operatorname{Re} K_{11} = \frac{1}{2R}\, \beta h(\alpha) \operatorname{Re}(-i\bar{\theta}), \quad h(\alpha) = \pi \operatorname{res}_\varepsilon \left(\frac{\bar{k}ps}{D}\right) +$$

$$\int_0^{\sqrt{\alpha}} \frac{\xi \sqrt{\alpha - \xi^2}\, d\xi}{\xi^2 \sqrt{(\alpha - \xi^2)(1-\xi^2)} + (\xi^2 - 1/2)^2} + \int_{\sqrt{\alpha}}^{1} \frac{\xi^3 (\xi^2 - \alpha) \sqrt{1-\xi^2}\, d\xi}{\xi^4 (\xi^2 - \alpha)(1-\xi^2) + (\xi^2 - 1/2)^2}.$$

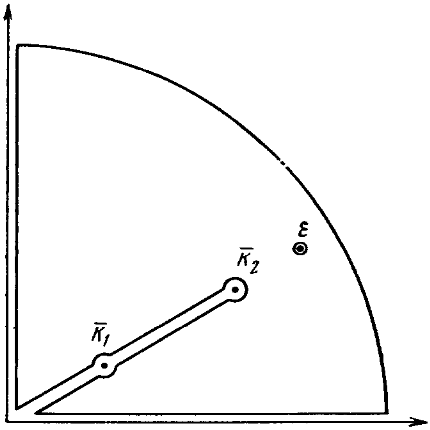

Fig. 1

We obtain analogously

$$\operatorname{Im} K_{11} = \frac{1}{2R}\, \beta h(\alpha) \operatorname{Im}(i\bar{\theta})$$

so that

$$K_{11} = \frac{1}{2R}\, \beta h(\alpha)\, i\theta$$

Substituting the estimate found into (4.2) and (4.4), we obtain

$$h_{11} = -h_{12} = \frac{\beta h(\alpha)\, i\theta}{2R\sqrt{1-\alpha^2}}\, x_1(r) \quad (4.5)$$

$$h_{21} = h_{22} = \frac{\beta h(\alpha)\, i\theta}{2R\sqrt{1-\alpha^2}}\, x_2(r)$$

Let us apply successive approximations to (4.3). Let us take the static solution (3.3) as the first approximation

$$\omega_1(r) = Ax_1(r), \quad \omega_2(r) = iAx_2(r)$$

By using the estimate (4.5) we obtain the first approximation solution of the system (4.3) as

$$\omega_1(r) = A_* x_1(r), \quad \omega_2(r) = iA_* x_2(r) \qquad (A_* = 1 + a\gamma h(\alpha)\, i\theta)$$

In a first approximation the solution of (2.7) has the form

$$\varphi_1(r) = 2A_* \cos\left(a \ln \frac{R-r}{R+r}\right), \qquad \varphi_2(r) = 2A_* \sin\left(a \ln \frac{R-r}{R+r}\right)$$

5. We find the reaction of the half-space (in the static case $\theta = 0$) by the methods of integrating multivalued functions

$$P = 2\pi \int_0^R \sigma_z(r, 0)\, r\, dr = 2\pi\mu_* \int_{-R}^{R} \varphi_1(t)\, dt =$$

$$4(\lambda_* + \mu_*)\,[1 + a\gamma h(\alpha)\, i\theta] \ln\left(\frac{\lambda_* + 3\mu_*}{\lambda_* + \mu_*}\right) b_0 R$$

Let us investigate the behavior of the stresses at the edge of the stamp. Let us note that because of the properties of $\varphi_1(t)$ and $\varphi_2(t)$ the relationships (2.2) can be represented as

$$\frac{1}{2\mu_*}\, \sigma_z = \frac{\varphi_1(r)}{\sqrt{R^2 - r^2}} - \int_r^R \frac{t\,[\varphi_1(t) - \varphi_1(r)]}{(t^2 - r^2)^{3/2}}\, dt$$

$$\frac{1}{2\mu_*}\, \tau_{rz} = \frac{\varphi_2(r)}{\sqrt{R^2 - r^2}} - \int_r^R \frac{r\,\varphi_2(t) - t\varphi_2(r)}{(t^2 - r^2)^{3/2}}\, dt$$

We set $t = R \operatorname{th} {}^1/_2\, \xi,\ r = R \operatorname{th} {}^1/_2\, x$, so that

$$\frac{1}{2\mu_*}\sigma_z = 2A \operatorname{ch}\frac{x}{2}\,[\cos(ax) + \varphi(x)], \qquad \varphi(x) = \int_x^\infty \frac{\psi_1(\xi, x)}{\psi_2(\xi, x)}\,d\xi$$

$$\psi_1 = \operatorname{th}\frac{\xi}{2}\sin\left(a\,\frac{\xi - x}{2}\right)\sin\left(a\frac{\xi + x}{2}\right)\left(\mathrm{c}\ \ \frac{x}{2}\right)^{1/2}$$

$$\psi_2 = \left(\operatorname{th}\frac{x}{2} + \operatorname{th}\frac{\xi}{2}\right)^{3/2}\left(\operatorname{sh}\frac{\xi - x}{2}\right)^{3/2}\left(\operatorname{ch}\frac{\xi}{2}\right)^{1/2}$$

The estimates

$$\left|\frac{\psi_1}{\psi_2}\right| < \frac{|a|}{2\sqrt{2}}\left(\operatorname{th}\frac{x}{2}\right)^{-3/2}\left(\operatorname{sh}\frac{\xi - x}{2}\right)^{-1/2}$$

$$|\varphi(x)| < \frac{|a|}{\sqrt{2}}\left(\operatorname{th}\frac{x}{2}\right)^{-3/2}\int_0^\infty (\operatorname{sh} u)^{-1/2}\,du < \frac{2\ln 3}{\pi}\left(\operatorname{th}\frac{x}{2}\right)^{-3/2}$$

are valid, hence $\varphi(x)$ is a continuous function and $|\varphi(x)| < 1$ starting with some x. Therefore, the equation $\cos(ax) + \varphi(x) = 0$ has an infinite number of zeros and the stress σ_z oscillates at the edge of the stamp. The oscillation of τ_{rz} is proved analogously. The same phenomenon evidently holds also in dynamics, at least for low vibration frequencies.

It should be noted that the method developed above is applicable for arbitrary Λ and M. However, the structure of $h(\alpha)$ is considerably more complex in the general case. In the particular case of an elastic medium α and θ take on only real positive values.

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ON THE PROBLEM OF THE RELATIONSHIP BETWEEN THE SCHWARZSCHILD AND TOLMAN METRICS

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The problem of the relationship between the Schwarzschild and Tolman metrics has occupied the attention of many workers. Although the solutions given in [1 – 3] satisfy the equations of the general relativity theory (OTO) (*), they contradict the correspondence principle. This means that for $G \to 0$, the interval is not transformed into the interval of the special relativity theory (CTO) (*), while for $c \to \infty$, the solutions do not become Newtonian. This is apparently caused by the unfortunate choice of the coordinates in the Tolman frame of reference. Papers [4, 5] illustrate particular cases of a correct passage from one metric to the other.

In the present paper a general method of obtaining solutions is proposed in which the passage from one frame of reference to the other satisfies the correspondence principle.

The intervals in the co-moving frame of reference and in the central frame of reference are, respectively,

$$-ds^2 = -c^2d\tau^2 + e^{\omega}dR^2 + r^2d\Omega^2 \qquad (1)$$

$$-ds^2 = -e^{\nu}c^2dt^2 + e^{\lambda}dr^2 + r^2d\Omega^2 \qquad (2)$$

$$(d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2)$$

Since $r = r(c\tau, R)$ and $ct = ct(c\tau, R)$, we have

$$dr = r^{\cdot}cd\tau + r'dR, \quad cdt = ct^{\cdot}cd\tau + ct'dR$$

$$(r^{\cdot} = \partial r / c\partial\tau, \ r' = \partial r / \partial R, \ ct^{\cdot} = c\partial t / c\partial\tau, \ ct' = c\partial t / \partial R)$$

Substituting these differentials into (2), equating the coefficients accompanying $c^2d\tau^2$ and dR^2 and remembering that the coefficient of $2cd\tau dR$ is zero, we obtain

$$e^{\nu}c^2t^{\cdot 2} - r^{\cdot 2}e^{\lambda} = 1, \qquad e^{\lambda}r'^2 - c^2t'^2e^{\nu} = e^{\omega}, \qquad e^{\lambda}r^{\cdot}r' - ct^{\cdot}ct'e^{\nu} = 0 \qquad (3)$$

from which, eliminating e^{λ} and e^{ν}, we have

$$e^{\lambda} = e^{\omega} / (r'^2 - r^{\cdot 2}e^{\omega}), \quad e^{\nu} = r'^2 / [c^2t^{\cdot 2}(r'^2 - r^{\cdot 2}e^{\omega})] \qquad (4)$$

$$(e^{\omega}ct^{\cdot}r^{\cdot} - ct'r')(ct^{\cdot}r' - ct'r^{\cdot}) = 0$$

*) Editors note. The abbreviations (OTO) and (CTO) are used in the relevant Soviet literature and stand for "general relativity theory" and "special relativity theory", respectively.